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Generalized Appell Connection Sequences. II

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1. INTRODUCTION

This is a continuation of [2], and familiarity with that paper is assumed. We recall that the main purpose of [2] was to show the connection sequence Q from one sequence in a given generalized Appell class (Φ) to another is itself in (Φ) and to exhibit a generalized Appell representation for Q. A special case of that representation was then translated into an operator method, due originally to Mullin and Rota [4], for finding connection sequences within a certain subclass of (exp).

In the present paper we translate the representation for Q in its full generality into the language of operators. Specifically, we show how Rota, Kahaner, and Odlyzko's [5] recent extension of Mullin and Rota's operator method to the entire class (exp) is, in fact, readily adapted to the problem of finding connection sequences within any generalized Appell class. Continuing then with the operator approach, we consider the problem of determining *subclasses* of (Φ) which always contain the connection sequence from P to $\tilde{P}$ when they contain P and $\tilde{P}$. The problem is suggested to us by the fact that this is the case for the important subclass of (exp) that Mullin and Rota considered.

Group-theoretic arguments, which we believe have considerable advantage in terms of conceptual clarity, play a central role here just as they did in [2]. Indeed, our results are essentially contained in the discussion of operator groups in Section 2 below, connection sequences being treated explicitly in Section 3.

2. SUBGROUPS OF THE SHEFFER GROUP

Following Rota, Kahaner, and Odlyzko, who concentrated on the class (exp), we assign to each sequence $P = \{P_n(x)\}_{n=0}^{\infty}$ in (Φ), with representation

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = G(t)\, \Phi(xH(t)) \tag{1}$$

involving the uniquely determined functions $H(t)$ and $G(t)$, the two differential operators

$$J(D) = H^{-1}(D) = \sum_{k=1}^{\infty} j_k D^k \qquad (j_1 \neq 0), \tag{2}$$

$$F(D) = \frac{1}{G(H^{-1}(D))} = \sum_{k=0}^{\infty} f_k D^k \qquad (f_0 \neq 0). \tag{3}$$

Here $D^k = d^k/dx^k$ and the function $H^{-1}(t)$ is the formal power series inverse of $H(t)$, defined by $H(H^{-1}(t)) = H^{-1}(H(t)) = t$. When operators

$$J(D) = \sum_{k=1}^{\infty} j_k D^k \quad (j_1 \neq 0) \qquad \text{and} \qquad F(D) = \sum_{k=0}^{\infty} f_k D^k \quad (f_0 \neq 0)$$

are given, the functions $H(t)$ and $G(t)$ in (1) are obtained by writing

$$H(t) = J^{-1}(t) \text{ and } G(t) = \frac{1}{F(J^{-1}(t))}. \tag{4}$$

Thus there is a one to one correspondence between ordered pairs of operators $[J(D), F(D)]$ and sequences P in (Φ).

If the pairs $[J(D), F(D)]$ and $[\tilde{J}(D), \tilde{F}(D)]$ correspond to sequences

$$P = \{P_n(x)\}_{n=0}^{\infty} \qquad \text{and} \qquad \tilde{P} = \{\tilde{P}_n(x)\}_{n=0}^{\infty},$$

respectively, the pair corresponding to the product $P\tilde{P} = \{(P\tilde{P})_n(x)\}_{n=0}^{\infty}$ can be written $[J(\tilde{J}(D)), F(\tilde{J}(D))\tilde{F}(D)]$. This was demonstrated by Rota, Kahaner, and Odlyzko [5, Theorem 7] in the case of (exp) and is immediate from [2] where we recall that if P and $\tilde{P}$ have representations (1) and

$$\sum_{n=0}^{\infty} \phi_n \tilde{P}_n(x)\, t^n = \tilde{G}(t)\, \Phi(x\tilde{H}(t)), \tag{5}$$

respectively, $P\tilde{P}$ has the representation

$$\sum_{n=0}^{\infty} \phi_n (P\tilde{P})_n\,(x)\, t^n = G(t)\, \tilde{G}(H(t))\, \Phi(x\tilde{H}(H(t))).$$

Evidently then, the operator pairs form a group under the binary operation

$$[J(D), F(D)] \cdot [\tilde{J}(D), \tilde{F}(D)] = [J(\tilde{J}(D)), F(\tilde{J}(D))\tilde{F}(D)], \tag{6}$$

the correspondence between them and the sequences in (Φ) being an isomorphism. For, according to [2], the class (Φ) can be regarded as a group whose products are $P\tilde{P} = \{(P\tilde{P})_n (x)\}_{n=0}^{\infty}$. Inasmuch as the class (exp) of Sheffer sequences is the prototype for (Φ), we refer to the above operator group as the *Sheffer group*. Note that its identity element is $[D, 1]$.

Two simpler and more obvious groups are formed when the operators $J(D)$ and $F(D)$ are taken separately. We let $\mathscr{J}$ denote the group of operators $J(D)$, or J, where multiplication is composition of formal power series: $(J \circ \tilde{J})(D) = J(\tilde{J}(D))$. We let $\mathscr{F}$, on the other hand, denote the (commutative) group of operators $F(D)$ where products are the usual Cauchy products: $(F\tilde{F})(D) = F(D)\tilde{F}(D)$. In view of (6), these two groups are identifiable as subgroups of the Sheffer group when $[J(D), 1]$ and $[D, F(D)]$ are written for $J(D)$ and $F(D)$, respectively. To be precise, $\mathscr{J}$ and the subgroup of pairs $[J(D), 1]$ are isomorphic, as are $\mathscr{F}$ and the subgroup formed by the pairs $[D, F(D)]$.

We note from (4) that the sequences in (Φ) to which the operator pairs $[J(D), 1]$ correspond are those with representations of the type

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = \Phi(xH(t)). \tag{7}$$

When $\Phi(t) = \exp t$, they are what Sheffer [6] originally called basic sequences; and so we call $\mathscr{J}$ the *basic subgroup* of the Sheffer group. It is also natural to refer to $\mathscr{F}$ as the *Appell subgroup* since the pairs $[D, F(D)]$ correspond to the sequences in (Φ) with representations

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = G(t)\, \Phi(xt), \tag{8}$$

those sequences being the Appell [1] sequences when $\Phi(t) = \exp t$.

Any subgroup of $\mathscr{J}$ or $\mathscr{F}$ can, of course, be interpreted as a subgroup of the Sheffer group. Rota, Kahaner, and Odlyzko pointed out, for example, that basic operators of the type $J(D) = aD/(cD + 1)$, $a \neq 0$, form a group under composition. So the operator pairs $[aD/(cD + 1), 1]$, $a \neq 0$, form a Sheffer subgroup. Pairs of the type $[D, (1 - D)^c]$ also form a Sheffer subgroup since the operators $F(D) = (1 - D)^c$ constitute a group within the subgroup of Appell operators.

We turn now to a decomposition of the Sheffer group which yields still other subgroups. Observe that each element J in $\mathscr{J}$ induces an automorphism $\alpha_J : F \leftrightarrows F^J$ of $\mathscr{F}$ where

$$\{F(D)\}\, \alpha_J = F(J(D)) \tag{9}$$

and that

$$\alpha_J \alpha_{\tilde{J}} = \alpha_{J \circ \tilde{J}}. \tag{10}$$

Hence we can construct a semi-direct product [3, p. 88] of $\mathscr{F}$ by $\mathscr{J}$ consisting of ordered pairs $[J, F]$ where multiplication is defined by the equation

$$[J, F] \cdot [\tilde{J}, \tilde{F}] = [J \circ \tilde{J}, F^{\tilde{J}} \tilde{F}]. \tag{11}$$

But (11) is simply (6) in different notation, and the above semi-direct product is therefore the Sheffer group. This is the main result of the section and can be stated as follows.

THEOREM. *The Sheffer group is a semi-direct product of the Appell subgroup by the basic subgroup, the associated automorphisms being indicated in* (9).

We can identify other subgroups via our decomposition by first considering the simpler problem of finding subgroups $\mathscr{J}'$ and $\mathscr{F}'$ of $\mathscr{J}$ and $\mathscr{F}$, respectively, and then verifying that α_J maps the operators in $\mathscr{F}'$ onto operators in $\mathscr{F}'$ for each J in $\mathscr{J}'$. If such is the case, the restrictions of these α_J's to $\mathscr{F}'$ will automatically be automorphisms of $\mathscr{F}'$, with property (10) holding; and the semi-direct product of $\mathscr{F}'$ by $\mathscr{J}'$ that arises will itself be a subgroup of the Sheffer group.

Consider, for example, the case when $\mathscr{J}'$ is the subgroup of basic operators of the type $aD/(cD + 1)$, $a \neq 0$. Letting $\mathscr{F}'$ be the entire Appell subgroup, we find that the operator pairs $[aD/(cD + 1), F(D)]$, $a \neq 0$, form a Sheffer subgroup. Note that any $\mathscr{J}'$ can be used when $\mathscr{F}' = \mathscr{F}$.

For another illustration, let $\mathscr{J}'$ denote the subgroup consisting of just the two basic operators D and $D/(D - 1)$; and let $\mathscr{F}'$ be the subgroup formed by Appell operators of the type $(1 - D)^c$. It follows that if we take all the operator pairs $[D, (1 - D)^c]$ together with all the pairs $[D/(D - 1), (1 - D)^c]$, we have a Sheffer subgroup.

3. Connection Sequences

Rota, Kahaner, and Odlyzko's operator method for finding connection sequences within (exp) was based on the fact that the polynomials $P_n(x)$ $(n \geqslant 0)$ of any Sheffer sequence P, with representation

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = G(t) \exp(xH(t)), \tag{12}$$

can be determined directly from the operator pair $[J(D), F(D)]$ corresponding to P. This is accomplished as follows.

First, the operator $J(D)$ is used in conjunction with one of Steffensen's [7] formulas to write the polynomials $B_n(x)$ $(n \geqslant 0)$ of the basic sequence B to which the pair $[J(D), 1]$ corresponds. The polynomials $P_n(x)$ then follow from the expression

$$P_n(x) = \frac{1}{F(D)} B_n(x) \qquad (n = 0, 1, 2, \ldots) \tag{13}$$

that Rota, Kahaner, and Odlyzko established [5, Proposition 1]. We note in passing that an efficient alternative to their derivation of (13) is to operate with $1/F(D)$ on each side of the representation

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \exp(xJ^{-1}(t)),$$

obtained from equations (4). For that yields

$$\sum_{n=0}^{\infty} \left\{\frac{1}{F(D)} B_n(x)\right\} \frac{t^n}{n!} = \frac{1}{F(J^{-1}(t))} \exp(xJ^{-1}(t)); \tag{14}$$

and, again by equations (4), the right-hand sides of (12) and (14) are equal.

We now show how this technique for obtaining Sheffer polynomials from their operators can actually be used as the basis of an operator method for finding connection sequences within any generalized Appell class. While ours is a natural extension of Rota, Kahaner, and Odlyzko's solution to the connection sequence problem within (exp), our presentation is fundamentally different from theirs in that it ultimately depends on the group-theoretic approach to generalized Appell representations which was initiated in [2].

We let $P = \{P_n(x)\}_{n=0}^{\infty}$ and $\tilde{P} = \{\tilde{P}_n(x)\}_{n=0}^{\infty}$ be arbitrary sequences in a given class (Φ), with representations (1) and (5), respectively; and we let $Q_n(x) = \sum_{k=0}^{n} q_{n,k} x^k$ $(n \geqslant 0)$ be the polynomials of the connection sequence Q from P to $\tilde{P}$. According to [2], the coefficients $q_{n,k}$ in each $Q_n(x)$ are related by the equation

$$q_{n,k} = \frac{k!}{n!} \frac{\phi_k}{\phi_n} c_{n,k} \tag{15}$$

to the coefficients $c_{n,k}$ in the polynomials $C_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$ $(n \geqslant 0)$ of the Sheffer sequence C whose representation is the special case of the representation

$$\sum_{n=0}^{\infty} \phi_n Q_n(x)\, t^n = \frac{G(t)}{(\tilde{G}\tilde{H}^{-1}(H(t)))} \Phi(x\tilde{H}^{-1}(H(t))) \tag{16}$$

for Q that occurs when $\Phi(t) = \exp t$. Thus, to find the $Q_n(x)$ it is sufficient to find the $C_n(x)$, which can be calculated once the operator pair for C, or Q, is known. That pair, of course, follows readily from representation (16) and definitions (2) and (3) of basic and Appell operators:

$$\left[H^{-1}(\tilde{H}(D)), \frac{\tilde{G}(D)}{G(H^{-1}(\tilde{H}(D)))}\right]. \tag{17}$$

Observe that if the sequences P and $\tilde{P}$ are specified via their operator pairs $[J(D), F(D)]$ and $[\tilde{J}(D), \tilde{F}(D)]$, respectively, rather than by their representations, equations (4) may be used to write (17) as

$$\left[J(\tilde{J}^{-1}(D)), \frac{F(\tilde{J}^{-1}(D))}{\tilde{F}(\tilde{J}^{-1}(D))}\right]. \tag{18}$$

This is the form of the solution to the connection sequence problem within (exp) that Rota, Kahaner, and Odlyzko gave [5, Corollary 4]. In that case equation (15) is bypassed since C itself is the connection sequence.

We consider now the main problem posed in Section 1, that of determining subclasses of (Φ) which contain all their connection sequences. In view of the expression $Q = P\tilde{P}^{-1}$ that was used in [2] for the connection sequence from P to $\tilde{P}$, it is easy to see that the desired subclasses are precisely the subgroups of (Φ) taken as a group. It then follows from the isomorphism between the Sheffer group and (Φ) that these subclasses are determined by the Sheffer subgroups.

Illustrations are provided by the Sheffer subgroups noted in Section 2, and we need only refer to equations (4) to write generalized Appell representations characterizing the subclasses. If, for example, P and $\tilde{P}$ both have representations of the type

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = \Phi\left(\frac{xat}{ct+1}\right) \qquad (a \neq 0),$$

then so does the connection sequence Q from P to $\tilde{P}$. Two other subclasses in which this happens are characterized by the representations

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = (1-t)^c\, \Phi(xt)$$

and

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = G(t)\, \Phi\left(\frac{xat}{ct+1}\right) \qquad (a \neq 0).$$

The same is true, moreover, of the subclass of all sequences $P = \{P_n(x)\}_{n=0}^{\infty}$ in (Φ) such that, for any value of c, either

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = (1-t)^c\, \Phi(xt)$$

or

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = (1-t)^c\, \Phi\left(\frac{xt}{t-1}\right).$$

Finally, we recall that representations (7) and (8) characterize the subclasses of (Φ) determined by the basic and Appell subgroups, respectively. Hence those subclasses also have the property we seek. It was Sheffer's basic sequences, occurring when $\Phi(t) = \exp t$ in (7), which Mullin and Rota concentrated on and which led us to the problem of discovering other subclasses of (exp), and (Φ), containing all their connection sequences.

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